Reduced-Order Models of Structures with Viscoelastic Components

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A useful method of modeling viscoelastic effects in structures for transient response analysis is to treat the frequency dependence of the usual damping model through the introduction of extra dissipation coordinates or internal variables. These new coordinates require a curve fit to the material loss factor data over a range of frequencies. Such a method has the disadvantage of introducing a large number of extra degrees of freedom that make response calculations and controller design very computationally intensive. Methods are examined to reduce the model size and, hence, ease this burden. Eigensystem truncation and balanced realizations are used to successfully reduce the full model. It is demonstrated that methods where the reduction transformation is based on the undamped model produce poor results. An iterative method to calculate the full eigensystem is introduced, using the frequency-dependent material modulus. In an alternative scheme, the model based on the physical degrees of freedom is reduced before the extra dissipation coordinates for the Golla–Hughes–McTavish method are introduced. Also considered is the possibility of reducing the model at the element level, before assembly, although this approach has difficulty retaining the damping characteristics of the full model.

I. Introduction

SEVERAL researchers have presented successful methods to model the effects of viscoelastic damping mechanisms that introduce hysterisis. An effective approach is to introduce additional coordinates or internal variables to account for the frequency-dependent and hysteretic behavior. Motivated by the need to produce finite element models that are capable of predicting the dynamic response of a structure or component, Golla and Hughes,[1] McTavish and Hughes,[2] Lesieutre and Mingori,[3] Lesieutre,[4] Lesieutre and Bianchini,[5] and Lesieutre and Lee[6] developed independent means of augmenting a finite element model with new coordinates containing damping properties found from material loss factor curves. Johnson et al.[7] start at the constitutive law level and characterize an internal variable method of modeling the viscoelastic effects, which are eventually used to augment a finite element model, greatly increasing its size. The Golla–Hughes–McTavish (GHM) method uses a second-order physical coordinate system, and the approach described in Refs. 3–6 uses a first-order state-space method called the augmenting thermodynamic fields (ATF) method. Both are superior to the modal strain energy (MSE) method proposed by Johnson et al.[8] Although the MSE method is substantially easier to use, both the ATF and the GHM methods are more accurate. These two more complex approaches are able to account for damping effects over a range of frequencies, complex mode behavior, transient responses, and both time- and frequency-domain modeling. Inman[9] applied the GHM approach to simple beams, and Banks and Inman[10] provided an alternate time-domain method for modeling hysterisis.

The ATF approach focuses on a first-order differential equation model of the viscoelastic behavior and results in an odd-order model rather then the usual second-order differential equation used in dynamic finite element models. However, it does allow the treatment of temperature effects.[11] The GHM approach on the other hand is well suited in the second-order form but is, as yet, unable to represent temperature effects. The retention of second-order form is

particularly important where the goal is to produce reduced mass, damping, and stiffness matrices for substructures or elements, as is the case in this paper. Therefore, the GHM approach is used exclusively, although the proposed reductions for the full model would be equally valid for the ATF approach.

II. GHM Modeling of Viscoelastic Components

The basic procedure of GHM is to start with plots of experimentally obtained complex modulus or loss factor data. These plots of modulus vs frequency are available from manufacturers of viscoelastic material and are curve fit to a rational polynomial. This rational polynomial, with coefficients reflecting the material properties of the test specimen, is next used to represent the Laplace transform of the hysteretic stress–strain relationship. The result is combined with the undamped model of the structure to produce a final model containing expanded coordinates and a damping matrix that captures the transient decay and complex mode behavior of the structure with the viscoelastic components.

The finite element model of a structure with viscoelastic components may be modeled as

$$\hat{M}\ddot{q} + \hat{D}\dot{q} + \hat{K}q + G\hat{K}_v q = \hat{f} \tag{1}$$

where G is the relaxation function of the viscoelastic. The displacement q and the applied force f are vectors, and $\hat{M}$, $\hat{D}$, and $\hat{K}$ are the mass, damping, and stiffness matrices of the elastic part of the structure. The matrix $\hat{K}_v$ is that part of the stiffness matrix that represents the viscoelastic material. Usually, most of the damping will come from the viscoelastic material, and so the damping matrix in Eq. (1) will be zero, that is, $\hat{D} = 0$. In the standard GHM method[1,12] the following expression in the Laplace domain for the viscoelastic material properties is assumed:

$$sG(s) = G_0 + h(s) = G_0 - \sum_{i=1}^{n} \frac{(d_i s + k_i)^2}{s^2 + \beta_i s + \delta_i} \tag{2}$$

where G_0, d_i, k_i, β_i, and δ_i are constants. A mass term could be included in the numerator of Eq. (2), but constraints are then required between the mass, damping, and stiffness terms.[12] Thus, it is most convenient to set the mass terms to zero from the outset, so that the structure and the auxiliary degrees of freedom used to model the viscoelastic material are coupled only through the damping and stiffness properties and not the inertia properties. This has the great

advantage of making the resulting mass matrix well conditioned. In general the modes predominantly involving the dissipation coordinates in Eq. (1) will be overdamped, but it is assumed that the real poles are paired so that the model may be written in the form of Eq. (2).

The equations of motion (1) may be written in the frequency domain as

$$[\hat{M}s^2 + \hat{D}s + \hat{K} + (G_0 + h(s))\hat{K}_v]q(s)$$
$$= \hat{f}(s) + \hat{M}(sq_0 + \dot{q}_0) + \hat{D}q_0 \qquad (3)$$

where q_0 and $\dot{q}_0$ are initial displacement and velocity vectors.

We now introduce the n auxiliary vectors z_i and create a model, in the Laplace domain, that is equivalent to Eqs. (2) and (3) as

$$[Ms^2 + Ds + K]x(s) = f + [M](sx_0 + \dot{x}_0) + [D]x_0 \qquad (4)$$

where

$$[M] = \begin{bmatrix} \hat{M} & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}, \qquad [D] = \begin{bmatrix} \hat{D} & d_1 R & \cdots & d_n R \\ d_1 R^T & \beta_1 I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_n R^T & 0 & \cdots & \beta_n I \end{bmatrix}$$

$$[K] = \begin{bmatrix} \hat{K} + G_0 \hat{K}_v & k_1 R & \cdots & k_n R \\ k_1 R^T & \delta_1 I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n R^T & 0 & \cdots & \delta_n I \end{bmatrix}, \qquad x(s) = \begin{Bmatrix} q(s) \\ z_1(s) \\ z_2(s) \\ \vdots \\ z_n(s) \end{Bmatrix}$$

$$x_0(s) = \begin{Bmatrix} q_0 \\ z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{Bmatrix}, \qquad \dot{x}_0(s) = \begin{Bmatrix} \dot{q}_0 \\ \dot{z}_{10} \\ \dot{z}_{20} \\ \vdots \\ \dot{z}_{n0} \end{Bmatrix}, \qquad f(s) = \begin{Bmatrix} \hat{f}(s) \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

and the matrix R is such that $RR^T = \hat{K}_v$. The initial conditions of the ith set of auxiliary degrees of freedom are given by z_{i0} and $\dot{z}_{i0}$. For a structure with no rigid-body modes the number of extra degrees of freedom added is n times the elastic degrees of freedom (DOF). If rigid-body modes are present, then the number of extra DOF may be reduced, and in this case R becomes a rectangular matrix. This extension to include rigid-body modes is relatively straightforward[1] and so will not be considered further. The initial conditions for the auxiliary degrees of freedom are[12]

$$z_{i0} = 0 \qquad (5)$$

$$\dot{z}_{i0} = -d_i R^T q_0 \qquad (6)$$

The parameters used in Eq. (2) as well as the number of terms used in the summation depend on the nature of the particular viscoelastic material used and the structure and are determined experimentally. Obviously, the GHM model has the potential for greatly increasing the order of the final model, hence motivating work on model reduction.

III. Calculating the Eigensystem of the Full Model

There are two ways of calculating the eigensystem of the full model. The GHM approach may be used to form the mass, damping, and stiffness matrices using Eq. (4). The eigensystem may be computed in the state space by rewriting the eigenvalue equation equivalent to Eq. (4) as

$$\lambda_i \begin{Bmatrix} x_i \\ \lambda_i x_i \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{Bmatrix} x_i \\ \lambda_i x_i \end{Bmatrix} \qquad (7)$$

where λ_i and x_i are the ith eigenvalue and eigenvector, respectively. These eigenvalues and eigenvectors will either be real or occur in complex conjugate pairs. Alternative, general forms of the general eigenvalue problem may be formulated to retain symmetric matrices. The biggest drawback with this approach is the size of the matrix and, hence, the computational effort required for the eigensolution. If three terms are required in the summation of Eq. (2) to successfully fit the viscoelastic material characteristics, then the number of DOF are approximately four times the number of physical DOF, requiring a computational effort approximately 64 times that required if the viscoelastic material modulus was constant.

An alternative method to compute the eigensystem is to apply an iterative scheme based on the assumed frequency-dependent complex modulus. Here the eigensolution is in terms of the physical coordinates, which from Eq. (1) is

$$\lambda_i \begin{Bmatrix} q_i \\ \lambda_i q_i \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -\hat{M}^{-1}[\hat{K} + \text{real}(j\omega G(j\omega))\hat{K}_v] & -\hat{M}^{-1}[\hat{D} + \text{imag}(j\omega G(j\omega))\hat{K}_v/\omega] \end{bmatrix} \begin{Bmatrix} q_i \\ \lambda_i q_i \end{Bmatrix} \qquad (8)$$

Notice that the real part of the viscoelastic modulus gives the stiffness properties, whereas the imaginary part contributes to the damping. The difficulty here is that the material properties of the viscoelastic depend on the frequency, which we do not know. However an iterative scheme may be set up as follows:

1) An estimate of the natural frequency is made (or the midpoint of the frequency range taken).

2) The material properties are obtained at this frequency, either from the GHM formula, Eq. (2), or by interpolation from the measured modulus data directly.

3) By the use of this value of modulus, the eigensystem is computed and the eigenvalue of interest is obtained.

4) If the eigenvalue has not converged, then the natural frequency is computed and steps 2 and 3 are repeated with this new frequency.

In practice convergence will be very fast. The size of each eigenproblem to be solved is much smaller than that of Eq. (7). However, an eigensolution must be obtained at every iteration, and the process must be repeated for each mode of interest. The resulting eigenvalues and eigenvectors will not be exactly the same as those obtained from Eq. (7), but they should be close. In practice the material modulus would be obtained from interpolation of the measured modulus data, thus avoiding the need to produce a GHM model of the material. Plouin and Balmès[13] considered a similar iterative scheme but used a basis of real modes, with corrections for the damping and static effects. An iterative version of the modal strain energy method may also be used, although the resulting damping estimates are only a first-order approximation to the full complex analysis for small damping.[14]

IV. Reduced-Order Models

For control design and analysis, the full model, given by Eq. (4), contains too many degrees of freedom. This model should be reduced to allow fast computation of the response, to test many candidate controller designs. This reduction is particularly valuable for structures with viscoelastic components because many of the modes that predominantly involve the dissipation coordinates will be overdamped modes, whereas the important modes are the underdamped structural modes, that include damping via the coupling in the damping and stiffness matrices. Two reduction methods will now be outlined. The reduction to modal coordinates is regularly used for undamped structures, through the system equivalent reduction expansion process (SEREP) transformation.[15, 16] The difference in this paper is that the complex eigenvectors must be used to incorporate the damping of the viscoelastic material.

The approach common in control engineering uses consideration of controllability and observability and transforms the states to balanced states that are equally controllable and observable.[17, 18] The least controllable and observable balanced states are then removed. The great advantage of the balanced realization approach is that the

important modes do not have to chosen a priori, and the overdamped modes are incorporated in a consistent way. The common approach to balanced realizations and model reduction is based on the state space, although the extension to second-order form, via a real transformation, has been considered by Meyer and Srinivasan.[19]

The time-domain, state-space model equivalent to Eq. (4) is

$$\dot{z} = Az + B\tilde{f}, \qquad \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} = Cz \qquad (9)$$

where

$$z = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}$$

$$B = \begin{bmatrix} \vdots \\ 0 \\ I \\ 0 \\ \vdots \end{bmatrix}, \qquad C = \begin{bmatrix} I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix}$$

Note that the input is assumed to be the force into the physical degrees of freedom, and the output is assumed to be the response (and velocity) at the physical DOF. Equation (9) represents the standard state-space model. It is possible to transform the states so that A is symmetric, but this has not been done here.

A. Reduction to Modal Coordinates

The preceding section has considered the calculation of the eigenvalues and eigenvectors of the structure. Equation (7) is equivalent to calculating the right eigenvectors ϕ_{Ri} and eigenvalues λ_i, so that

$$A\phi_{Ri} = \lambda_i \phi_{Ri} \qquad (10)$$

The left eigenvectors ϕ_{Li} may also be defined as

$$\phi_{Li}^T A = \lambda_i \phi_{Li}^T \qquad \text{or} \qquad A^T \phi_{Li} = \lambda_i \phi_{Li} \qquad (11)$$

Once the modes of interest have been chosen, then the left and right eigenvectors may be arranged as columns in two matrices, Φ_L and Φ_R. Generally, these matrices are rectangular, and only the modes that have significant influence in the frequency range of interest are retained. Provided that the eigenvectors are normalized so that $\phi_{Li}^T \phi_{Ri} = 1$, the state-space matrices may be transformed to give

$$\dot{z}_e = A_e z_r + B_e \tilde{f}, \qquad \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} = C_e z_e \qquad (12)$$

where

$$A_e = \Phi_L^T A \Phi_R, \qquad B_e = \Phi_L^T B, \qquad C_e = C \Phi_R$$

These equations are not directly useful because the reduced state vector is complex, as are the reduced matrices. Furthermore, we would like the equations in terms of the physical DOF q and their derivatives. Assume that the number of states and the number of eigenvectors of interest are equal, so that C_e is square. Furthermore, the modes may be are chosen so that C_e is well conditioned with respect to inversion. If C_e is singular or poorly conditioned, then some of the chosen modes have little effect on the displacement at the physical DOF. The reduced model based on the displacement and velocity at the physical DOF may then be formed:

$$\begin{Bmatrix} \dot{q}_r \\ \ddot{q}_r \end{Bmatrix} = A_r \begin{Bmatrix} q_r \\ \dot{q}_r \end{Bmatrix} + B_r \tilde{f} \qquad (13)$$

where

$$A_r = C_e A_e C_e^{-1}, \qquad B_r = C_e B_e$$

Although it is not immediately apparent that the matrices in Eq. (13) are real, the inclusion of complex conjugate pairs of eigenvectors ensures that this is the case. If there are fewer physical DOF

than modes, then a reduced model may be produced, where the displacement and velocity at the physical DOF form a subset of the states.

Although Eq. (13) implies that the original matrices must be used, this is not the case. Indeed, knowledge of the normalized eigenvectors and eigenvalues to be retained is sufficient to generate the reduced model. From the definition of A_e and the eigenvalue equations,

$$A_e = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_r}) \qquad (14)$$

where n_r is the number of retained eigenvalues. From the definition, Eq. (12), B_e and C_e are obtained by picking out the DOF of interest from the eigenvector matrices. Thus, the model given in Eq. (12) may be computed directly from measurements or from a reduced modal model computed via the iterative approach. Note that in the case of the eigensystem computed by the iterative approach given earlier Eq. (14) is approximate, because the eigenvectors are truly normalized individually. Because the state-space matrix [Eq. (8)] changes for each eigenvalue, the eigenvectors corresponding to different eigenvalues are not guaranteed to be orthogonal.

B. Reduction via Balanced Realizations

A system is controllable if an input exists so that the states of the system may be driven to any arbitrary configuration. The controllability gramian W_c for the system described by Eq. (9) is the solution of the equation

$$AW_c + W_c A^T + BB^T = 0 \qquad (15)$$

The system is controllable if the gramian is full rank, and the condition of the gramian is a measure of the controllability of the system. Similarly, the observability gramian W_o may be obtained from the solution of

$$A^T W_o + W_o A + C^T C = 0 \qquad (16)$$

If the system is observable, then the state vector may be reconstructed from knowledge of the current and past values of the output vector.

The degree of controllability and observability cannot be determined in isolation.[17] A convenient approach is to transform the state so that the observability and controllability gramians are equal. This is a balanced realization. It is also convenient to make these gramians diagonal, so that the controllability and observability of individual states may be determined immediately. The least controllable and observable transformed states may then be removed. Skelton[17] outlines the method and gives a convenient algorithm. Meyer and Srinivasan[19] considered the balanced realization based on the second-order form of the equations of motion (1) or (4) and gave a method of calculating the balancing transformation for the degrees of freedom. In state-space form, if T_b is the required balancing transformation, then Eq. (9) becomes

$$\dot{z}_b = A_b z_b + B_b \tilde{f}, \qquad q = C_b z_b \qquad (17)$$

where

$$z = T_b z_b, \qquad A_b = T_b^{-1} A T_b, \qquad B_b = T_b^{-1} B, \qquad C_b = C T_b$$

The balanced states are ordered from most controllable and observable to least, so the state vector and the matrices may be partitioned into retained and eliminated states. On the assumption that the derivative of the slave states is zero, then a reduced state-space model based solely on the retained states may be formed.[17] Calculating balanced realizations is computationally intensive. Furthermore, methods that generate the transformation using a Cholesky decomposition often fail for systems with a large number of states, because the gramians are rank deficient to the numerical precision used in typical software packages. Skelton[17] suggests removing uncontrollable states before calculating the observability gramian. This can be taken further, and many of the least controllable states could be removed at this stage. Sufficient states should be kept to ensure that there are enough observable states to generate the reduced model.

For the large-scale systems in structural dynamics, the initial model will also have to be reduced to an intermediate-size model using eigensystem truncation, described earlier or one of the standard methods for undamped structures, for example, static reduction,[20] improved reduced system (IRS) model,[21] or SEREP.[15,16]

V. Reduction at the Substructure or Element Level

The major problem with using the full, large-scale model to generate the reduced model is that this model will have a very large number of DOF. This section introduces two approaches that could potentially save substantial computational effort. The first approach is to assemble the model of the physical parts of the structure [that is, assemble the matrices in Eq. (1)] and reduce this model before the GHM dissipation coordinates are introduced. This reduction could be undertaken for a substructure containing the viscoelastic material, and this substructure is then combined with the remaining substructures in the usual way. This approach reduces the number of physical coordinates and also the number of dissipation coordinates. Any suitable reduction of the physical DOF may be used, for example, static reduction, dynamic reduction, IRS, SEREP or balanced realizations. Because the viscoelastic material adds most of the damping, standard methods for undamped structures perform well, and in the numerical example given later SEREP is used.

The second approach is to reduce the models of the elements before assembly. The difference with the first approach is the order in which the reduction and GHM formulation are undertaken, and therefore, the DOF in the resulting reduced model will have different characteristics. In the substructure approach the DOF include dissipation coordinates, whereas in the reduction at element level only physical DOF remain. Both approaches will be demonstrated in the numerical example, but some of the problems in the reduction at element level will now be discussed. The auxiliary DOF in Eq. (4) now have to be included for each element, and these DOF are the obvious candidates to be eliminated to produce the reduced element model. Some of the physical DOF may also be reduced out of the element model. Methods used for undamped systems, such as static reduction, may be used, but the structure of the element matrices [Eq. (4)] means that these methods do not retain the damping properties of the element. With static reduction, for example, this is because the reduction is equivalent to enforcing a viscoelastic material model that is quadratic in frequency. It is possible to use a truncated complex eigensystem, and providing the correct modes are used, Eq. (13) may be applied. The resulting reduced state-space matrix does have the correct form for a second order system (namely, the upper half is a response and the lower half is the corresponding velocity), and so reduced mass, damping, and stiffness matrices may be obtained. Even so, as shown in the example, there are still problems in retaining the damping effectiveness in the assembled model. Balanced realizations may be used to generate reduced models of the element in second-order form.[19] A transformation based on the undamped modes must be undertaken, so that the rigid-body modes are decoupled from the structural modes, to ensure that the model is asymptotically stable and, thus, meets the requirements for the application of the balancing approach. The resulting model produces frequency responses for the retained physical DOF that are very accurate. However the resulting assembled model has poor accuracy, as demonstrated in the example.

VI.   Numerical Example

The methods outlined were tested on the cantilever beam shown in Fig. 1, whose dimensions are given in Table 1. The beam consisted of an aluminum base beam, with a layer of 3M ISD 112 viscoelastic material, followed by an aluminium constraining layer. Only the base beam was clamped. The finite element model used the formulation given by Lesieutre and Lee[6] except that the viscoelastic was modeled using the GHM approach with three terms in the summation of Eq. (2) (Ref. 12). The element proposed by Lesieutre and Lee has 10 DOF per element. Each node has four DOF, namely, the three standard displacements of the base beam (axial displacement and bending transverse displacement and rotation) and the shear angle in the viscoelastic. Each element has two internal DOF, namely, an

Table 1   Parameters of the cantilever beam[a] example

Parameter	Base beam	Viscoelastic layer	Constraining layer
Young's modulus, GN/m^2	70	ISD 112	70
Mass density, kg/m^3	2700	1600	2700
Thickness, mm	5	0.25	1
Poisson's ratio	——	0.4	——

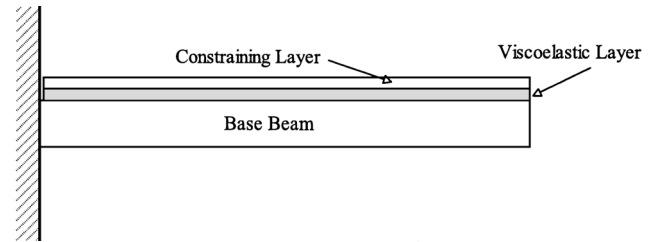[a]Beam length 200 mm, beam width 10 mm.

Fig. 1   Example cantilever beam with constrained layer damping (base beam and constraining layer are aluminium, the viscoelastic is ISD 112).
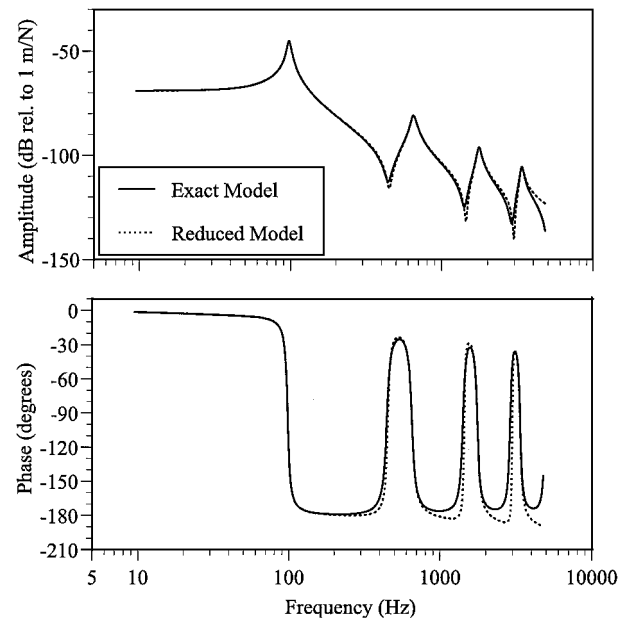

Fig. 2   Comparison of the FRF for the full model and the model reduced by eigensystem truncation.

axial displacement and a viscoelastic shear angle. The element has three rigid-body modes. Thus, in a 10-element model, to be used to demonstrate the reduction methods, there are 61 physical DOF and a further 183 auxiliary DOF, giving a total of 244 DOF.

For the beam with 10 elements, Table 2 shows the natural frequencies and damping ratios of the beam that are within the frequency range of the viscoelastic model (between 10 and 4800 Hz) obtained for the full model using the GHM approach and also using the iterative approach. In practice the iterative approach would obtain the material properties by interpolation of the measured data, but here the GHM expression for the material properties was used to enable the comparison of the methods. It is clear that the methods did not give exactly the same eigenvalues because during the iterative process a real frequency must be generated from the complex eigenvalue to estimate the complex material properties. However, the natural frequencies and damping ratios are very close. Figures 2 and 3 compare the frequency response function, where the beam was forced at the tip and the response was measured at the tip, for the modal truncation methods and the full model. Only the first four modes (eight eigenvalues) are used in the modal truncation approach, although the two smallest real eigenvalues were

**Table 2   Natural frequencies and damping ratios (in parentheses) for the cantilever beam
with constrained layer damping: reduction of the full model**

| Full model, 244 DOF	Iterative method	Balanced realization, full model reduced to 12 states	Static reduction to the 61 physical DOF
Underdamped modes, Hz (%)			
98.224 (2.83)	98.156 (2.81)	98.224 (2.83)	99.072 (7.45)
652.42 (4.17)	651.36 (4.15)	652.39 (4.17)	713.10 (6.47)
1757.6 (3.27)	1756.0 (3.25)	1757.9 (3.29)	2060.8 (7.81)
3361.0 (2.61)	3359.4 (2.58)	3362.5 (2.20)	4097.8 (8.57)
5490.3 (2.12)		5494.5 (2.20)	6237.1 (0.91)
Real poles			
−66.351		−248.23	−108.01
−243.31		−2223.9	−410.75
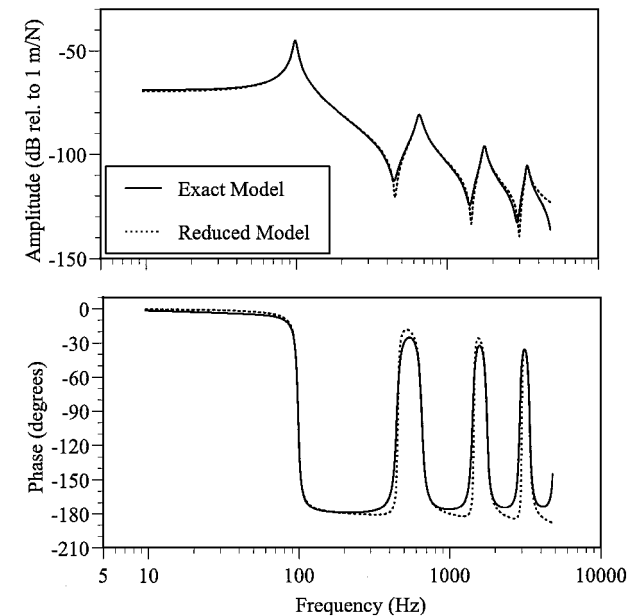
Fig. 3   Comparison of the FRF for the full model and the model reduced
by eigensystem truncation calculated by the iterative method.

Fig. 4   Comparison of the FRF for the full model and the model reduced
by balanced realizations.

also included to improve the low-frequency correlation. These real
eigenvalues were not available from the iterative approach, leading
to larger errors in the low-frequency range.

Also shown in Table 2 are the eigenvalues for a balanced real-
ization of the full model reduced to the 12 states. An intermediate
model containing the 64 smallest real poles and the lowest 6 un-
damped modes was used to ease the computation of the balanced
realization. The real poles include all of those lower in magnitude
than the first three underdamped eigenvalues. This initial reduction
is somewhat arbitrary, but should produce an intermediate model
that is small enough to allow the calculation of the balancing trans-
formation, but large enough to incorporate all of the important dy-
namics of the structure. A model reduced to 10 states was tried,
but produced a poor fit to the full model. Figure 4 compares the
frequency response function for this approach with that of the full
model. Obviously, the reduced model is well able to represent the
full model in the frequency range of interest.

Static reduction of the full model was also tried, and Table 2
shows the resulting natural frequencies and damping ratios when
the model is reduced to the physical degrees of freedom. After the
dissipation coordinates are removed, the viscoelastic material prop-
erties cannot be incorporated into the full model in an accurate
way, leading to considerable errors in both natural frequencies and
damping. It may easily be shown from Eq. (4) that this reduction
models the frequency dependence of the viscoelastic material by
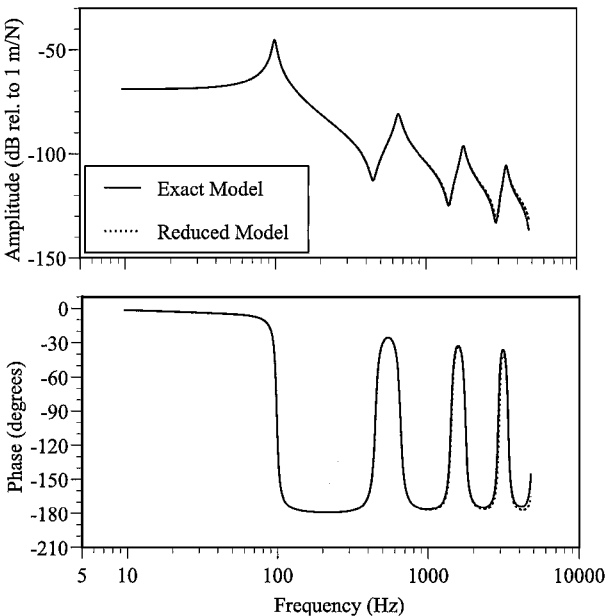a quadratic function, leading to considerable errors in the material
modulus.

Suppose that the physical model of the beam is assembled to give
the matrices in Eq. (1). The beam model is reduced using a SEREP
transformation based on the modes of the undamped structure with
an assumed viscoelastic modulus of 1 MN/m². This transforma-
tion is used to reduce all of the matrices in Eq. (1), which now
includes the frequency-dependent viscoleastic modulus. Therefore,
there is some loss of accuracy in the reduction transformation over
structures with a constant modulus. The GHM approach is now
applied to this reduced model, to give a reduced model including
dissipation coordinates to model the viscoelastic material properties.
Table 3 shows the natural frequencies and damping ratios obtained
when the beam model is reduced to three, two, and one DOF per
node, where u, w, and w' are the axial displacement, the transverse
displacement, and the beam rotation, respectively. Clearly, a signifi-
cant reduction in the number of DOF is possible without a significant
loss of accuracy in the frequency range of interest.

It was then attempted to reduce the element matrices before as-
sembly. Once again three terms were used in Eq. (2) to model the
viscoelastic material using the GHM approach. The element had 10
DOF and 3 rigid-body modes. Thus, there were 21 auxiliary DOF,
giving 31 DOF in total for the element including the viscoelastic.
Only the 10 physical DOF per element were retained. Static re-
duction, a truncated eigensystem, and a balanced realization were
tried. For the eigensystem truncation only the underdamped modes
were retained. Table 4 shows the estimated natural frequency and
damping ratio of the assembled beam obtained using these methods.
It is clear that the models of the assembled beam give significant

Table 3 Natural frequencies and damping ratios (in parentheses) for the cantilever beam with constrained layer damping: reduction of structural matrices before the application of the GHM methodology

Full model, 61 physical DOF	SEREP reduction to 3 DOF per node u, w, w'	SEREP reduction to 2 DOF per node u, w	SEREP reduction to 1 DOF per node w
		Underdamped modes, Hz (%)	
98.224 (2.83)	98.248 (2.83)	98.212 (2.84)	98.143 (3.02)
652.42 (4.17)	652.48 (4.17)	652.46 (4.22)	655.02 (7.98)
1757.6 (3.27)	1757.8 (3.27)	1760.6 (3.66)	1769.4 (5.00)
3361.0 (2.61)	3361.6 (2.63)	3367.2 (3.13)	3372.0 (3.49)
5490.3 (2.12)	5494.3 (2.25)	5496.5 (2.42)	5502.6 (2.66)
		Real poles	
−66.351	−66.379	−66.547	−104.14
−243.31	−243.42	−244.16	−321.73
		Physical DOF	
61	30	20	10
		Fictitious DOF	
183	48	48	27
		Total No. DOF	
244	78	68	37

Table 4 Natural frequencies and damping ratios (in parentheses) for the cantilever beam with constrained layer damping: reduction of element matrices by reducing out fictitious or dissipative DOF

Full model, 61 physical DOF	Static reduction	Eigensystem truncation	Balanced realization
		Underdamped modes, Hz (%)	
98.224 (2.83)	99.072 (7.47)	109.73 (0.00)	100.06 (5.93)
652.42 (4.17)	713.10 (6.47)	697.91 (0.01)	714.21 (5.95)
1757.6 (3.27)	2060.8 (7.81)	1866.9 (0.05)	2066.4 (7.55)
3361.0 (2.61)	4097.8 (8.57)	3506.3 (0.08)	4132.0 (8.65)
5490.3 (2.12)	6237.1 (0.91)	5633.2 (0.11)	6224.7 (0.64)
		Real poles	
−66.351	−12.409		−28.593
−243.31	−108.01		−139.55

Table 5 Natural frequencies and damping ratios for the underdamped eigenvalues for the element matrices[a]

Full model, kHz (%)	Static reduction, kHz (%)	Eigensystem truncation, kHz (%)	Balanced realization, kHz (%)
40.49 (8.00)	20.42 (99.87)	40.49 (8.00)	−179.65/−2.0384e5
63.71 (0.12)	45.88 (45.05)	63.71 (0.12)	63.81 (−0.02)
122.5 (0.02)	79.64 (0.90)	122.5 (0.02)	122.6 (0.00)
185.4 (0.02)	138.1 (0.38)	185.4 (0.02)	185.5 (0.00)
263.6 (0.03)	209.7 (0.80)	263.6 (0.03)	264.0 (0.00)
274.6 (0.00)	301.1 (0.49)	274.6 (0.00)	274.7 (0.00)
610.7 (0.00)	−77.969/−2.6455e5	610.7 (0.00)	610.9 (0.00)

[a]Three rigid-body modes.

errors in natural frequency and very poor damping estimates. Static reduction gives the same model irrespective of whether the reduction is performed at the element level or for the assembled beam. Table 5 compares the natural frequencies and damping ratios of the element models and shows good agreement for balanced realization. Obviously for the truncated eigensystem they are exact. Figure 5 shows the frequency response function (FRF) for the element, where a translational force was applied at one end of the beam, and the response was measured at the same point. Both the full element model and the model based on the truncated eigensystem are shown. Figure 6 shows the equivalent response comparison for the balanced realization. Obviously, the agreement in both cases is very good. It is thought that the poor quality assembled model arises because the dissipation coordinates are reduced out, leading to significant errors during assembly because of the frequency-dependent material modulus. Essentially, the constraints between the dissipation coordinates between elements cannot be enforced.

A further difficulty is that the element model is reduced based on the free–free eigensystem. In this configuration little shear is produced in the viscoelastic layer, leading to very lightly damped modes in the assembled model. The reduction based on a balanced realization uses the free–free input–output relationship, and although there are two real modes in the reduced model, Table 5 shows that some of the very lightly damped models become unstable when the model is truncated. Attempts by the authors to base the reduction on a combination of rigid-body modes and constrained modes (for example, clamped–free) have thus far been unsuccessful.

VII. Computational Issues

This paper has considered a number of reduction methods to produce lower-order models of structures with viscoelastic components. The goal is to produce a model that accurately represents the structure but may be used to efficiently design control systems or simulate the transient response of the structure. Thus, much of the computational effort would occur after the generation of the reduced model. However, it is still worthwhile to consider the computational effort required to produce the reduced models described. This will take the form of a discussion, rather than adding a CPU time or
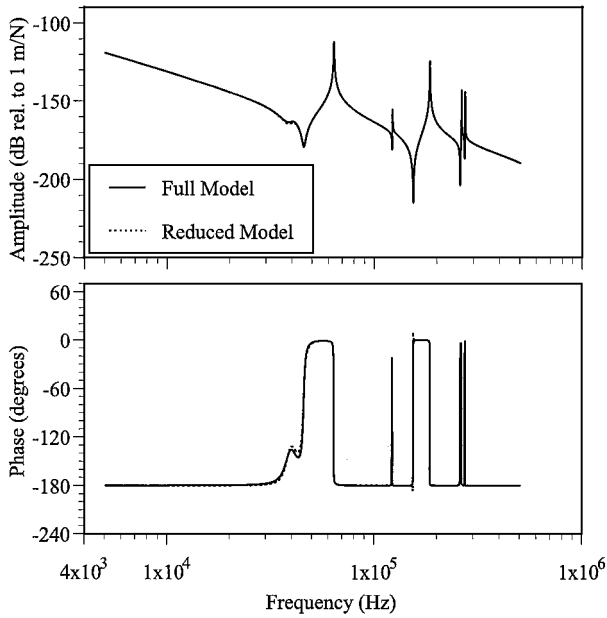
Fig. 5 Comparison of the FRF for a single element and the model reduced by eigensystem truncation.
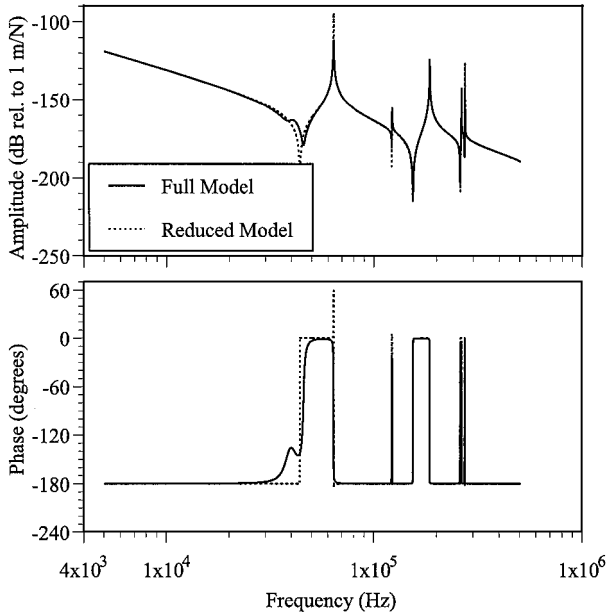

Fig. 6 Comparison of the FRF for a single element and the model reduced by balanced realization.

number of floating-point operations row to the tables of results, because there are a number of factors (described later) that make such a numerical comparison of limited value.

Reducing the full model using eigensystem truncation requires the calculation of the eigenvalues and eigenvectors of the state-space matrix, which is proportional to n^3, where n is the size of the matrix. The constant of proportionality is dependent on the method used to compute the eigensystem, how many eigenvalues are required, the bandwidth of the matrices, which form of the state-space matrix is used, and so on. The iterative approach to calculating the eigensystem requires the calculation of a larger number of eigensystems, but only at the physical DOF. In the numerical example there are 61 physical DOF and 244 DOF in total. Therefore, each eigensystem computation in the iterative approach requires a factor of $4^3 = 64$ less computation. Against this, there are four eigenvalues in the frequency range, and if each required four iterations for convergence (this is more than adequate in the example, but the number of iterations invariably depends on the specified convergence tolerance) then the saving in computation for the iterative approach is a factor of 4. Further saving may be made by interpolating the material prop-

erties so that the optimization of the GHM model to the measured data does not have to be undertaken. The reduction via a balanced realization requires an intermediate model based on eigensystem truncation, and therefore requires a similar computational effort to the straight eigensystem truncation.

In the reduction of the physical substructure before the application of the GHM methodology, there are two eigensystems to be computed. The first is based on the undamped model to generate the SEREP reduction of the physical substructure. Because this is based on the undamped model, in the example the size of the matrices from which the eigensystem is computed is a factor of 8 smaller than the full state-space matrix, and furthermore the eigensystem will be real. Once the dissipation coordinates have been added to the reduced model, the eigensystem must be calculated. Table 3 gives the number of DOF in these reduced models and even retaining 3 DOF per node produces a computational saving of a factor of $(\frac{244}{78})^3 \approx 31$.

Reduction at the element level is very computationally efficient. At the element level the matrices are very small and either the eigensystem or the balanced realization may be computed quickly. Furthermore, in the numerical example, the assembled model has only 61 DOF. Unfortunately, the poor quality of the reduced model would not allow this approach to be used in practice.

VIII. Conclusions

This paper has considered the possibility of reducing the order of finite element models of structures with viscoelastic components. Both the eigensystem truncation method and the balanced realization approach are able to reduce the order of the full model but still retain the important dynamic properties. These methods applied to the full model were successful and produced an accurate reduced-order model. Methods using reduction transformations based on the undamped model performed poorly. A new iterative method to calculate the eigensystem of the full structure was introduced that generated an accurate reduced model with some potential saving in computation and avoiding the need to generate a GHM model for the viscoelastic material. It was demonstrated that the structural model may be reduced before the GHM approach (and, therefore, the extra dissipation coordinates) is applied, and this gives a reasonably accurate damped model. Reducing the matrices at the element level back to the physical DOF, after the GHM terms have been included, produced a poor damped model of the structure. It is postulated that this poor performance is because, when the dissipation coordinates are removed, the viscoelastic material properties cannot be incorporated into the full model during the assembly process in an accurate way. This is disappointing, because such reduced models at the element level would enable components containing viscoelastic material to be incorporated easily into standard finite element codes.

References

[1]Golla, D. F., and Hughes, P. C., "Dynamics of Viscoelastic Structures—A Time Domain, Finite Element Formulation," *Journal of Applied Mechanics*, Vol. 52, No. 4, 1985, pp. 897–906.

[2]McTavish, D. J., and Hughes, P. C., "Modeling of Linear Viscoelastic Space Structures," *Journal of Vibration and Acoustics*, Vol. 115, No. 1, 1993, pp. 103–113.

[3]Lesieutre, G. A., and Mingori, D. L., "Finite Element Modeling of Frequency-Dependent Material Properties Using Augmented Thermodynamic Fields," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 6, 1990, pp. 1040–1050.

[4]Lesieutre, G. A., "Finite Elements for Dynamic Modeling of Uniaxial Rods with Frequency Dependent Material Properties," *International Journal of Solids and Structures*, Vol. 29, No. 12, 1992, pp. 1567–1579.

[5]Lesieutre, G. A., and Bianchini, E., "Time Domain Modeling of Linear Viscoelasticity Using Anelastic Displacement Fields," *Journal of Vibration and Acoustics*, Vol. 117, No. 4, 1995, pp. 424–430.

[6]Lesieutre, G. A., and Lee, U., "A Finite Element for Beams Having Segmented Active Contrained Layers with Frequency-Dependent Viscoelastics," *Smart Materials and Structures*, Vol. 5, No. 5, 1996, pp. 615–627.

[7]Johnson, A. R., Tessler, A., and Dambach, M., "Dynamics of Thick Viscoelastic Beams," *Journal of Engineering Materials and Technology*, Vol. 119, No. 3, 1997, pp. 273–278.

[8]Johnson, C. D., Keinholz, D. A., and Rogers, L. C., "Finite Element Prediction of Damping in Beams with Constrained Viscoelastic Layers," *Shock and Vibration Bulletin*, Vol. 51, No. 1, 1981, pp. 71–81.

[9]Inman, D. J., "Vibration Analysis of Viscoelastic Beams by Separation of Variables and Modal Analysis," *Mechanics Research Communications*, Vol. 16, No. 4, 1989, pp. 213–218.

[10]Banks, H. T., and Inman, D. J., "On Damping Mechanisms in Beams," *Journal of Applied Mechanics*, Vol. 58, No. 3, 1991, pp. 716–723.

[11]Lesieutre, G. A., and Govindswamy, K., "Finite Element Modeling of Frequency-Dependent and Temperature-Dependent Dynamic Behavior of Viscoelastic Material in Simple Shear," *International Journal of Solids and Structures*, Vol. 33, No. 3, 1996, pp. 419–432.

[12]Friswell, M. I., Inman, D. J., and Lam, M. J., "On the Realization of GHM Models in Viscoelasticity," *Journal of Intelligent Material Systems and Structures*, Vol. 8, No. 11, 1997, pp. 986–993.

[13]Plouin, A.-S., and Balmès, E., "Pseudo-Modal Representations of Large Models with Viscoelastic Behavior," *Proceedings of the 16th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1998, pp. 1440–1446.

[14]Wilson, C. J., Carnevali, P., Morris, R. B., and Tsuji, Y., "Viscoelastic Damping Calculations Using a p-Type Finite Element Code," *Journal of Applied Mechanics*, Vol. 59, No. 3, 1992, pp. 696–699.

[15]Kammer, D. C., "Test-Analysis-Model Development Using an Exact Modal Reduction," *International Journal of Analytical and Experimental Modal Analysis*, Vol. 2, No. 4, 1987, pp. 174–179.

[16]O'Callahan, J. C., Avitabile, P., and Riemer, R., "System Equivalent Reduction Expansion Process (SEREP)," *Proceedings of the 7th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1989, pp. 29–37.

[17]Skelton, R. E., *Dynamic Systems Control: Linear Systems Analysis and Synthesis*, Wiley, New York, 1988.

[18]Inman, D. J., *Vibration: With Control, Measurement and Stability*, Prentice–Hall International, 1989, pp. 141–182.

[19]Meyer, D. G., and Srinivasan, S., "Balancing and Model Reduction for Second-Order Form Linear Systems," *IEEE Transactions on Automatic Control*, Vol. 41, No. 11, 1996, pp. 1632–1644.

[20]Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, p. 380.

[21]O'Callahan, J. C., "A Procedure for an Improved Reduced System (IRS) Model," *Proceedings of the 7th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1989, pp. 17–21.

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